

The University of Chicago

STELLAR MODELS

A DISSERTATION SUBMITTED TO THE FACULTY
OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF DOCTOR
OF PHILOSOPHY

DEPARTMENT OF ASTRONOMY AND ASTROPHYSICS
1947

By
MARJORIE HALL HARRISON

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THE GENERALIZED COWLING MODEL

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Received July 28, 1944

ABSTRACT

Stellar models are considered in which the ratio of the mean molecular weight in the core to the mean molecular weight in the envelope varies from 1 to 2. It is found that, as μ_c/μ_e increases, the fraction of the mass and radius inclosed in the convective core decreases, while the total radius and luminosity of the configuration increases.

In examining the evolution of the main-sequence stars it appears useful to consider stellar models in which the convective core and the point-source envelope have different mean molecular weights; for, as the nuclear reaction takes place predominantly only in the convective core, we may expect the mean molecular weight of this region gradually to increase. With an increase in the molecular weight, we can no longer let the mean molecular weight μ_c , in the core be equal to μ_e , the mean molecular weight in the envelope.

In an investigation carried out by M. Schonberg and S. Chandrasekhar¹ two values of the ratio μ_c/μ_e were considered, namely, $\mu_c/\mu_e = 1$ (the standard Cowling model) and $\mu_c/\mu_e = 2$. It is now proposed to determine the effect of intermediate values on the fraction of the mass and radius inclosed in the convective core and the total radius and luminosity of the configuration.

We shall consider a composite model made up of a central core described by the Lane-Emden function of index $n = \frac{3}{2}$ and a point-source envelope. In constructing this model the method used was that proposed by Chandrasekhar;² in such a configuration the values of the temperature, pressure, and mass of the core should be identical at the interface, along with the condition that the effective polytropic index n attain the value 1.5 at the interface. We then have

$$P(r_i)_c = P(r_i)_e; \quad T(r_i)_c = T(r_i)_e; \quad M(r_i)_c = M(r_i)_e; \quad n = 1.5, \quad (1)$$

where P , M , and T denote the total pressure, the mass within the radius r , and the temperature, respectively; c refers to the core, e to the envelope, and i to the interface.

Introducing the homology invariant quantities u and v defined as in Chandrasekhar's *Stellar Structure* (pp. 146 ff.), we have for the conditions at the interface:

$$u_i = \frac{4\pi \rho_e(r_i) r_i^3 \mu_c}{M_e(r_i) \mu_e}; \quad v_i = \frac{2}{5} \frac{GM_e(r_i) \rho_e(r_i) \mu_c}{r_i P_e(r_i) \mu_e}. \quad (2)$$

The Fairclough³ solutions of the Lane-Emden equation of index $\frac{3}{2}$ were used for the regions of the star inside $r = r_i$. The regions outside $r = r_i$ will be governed by the same equations as for the point-source model with negligible radiation pressure. Seven integrations (due to Miss I. Nielsen) for the point-source model with solar values of L , M , and R , and $\mu = 1$ and $\log \kappa_0 = 24.992, 24.892, 24.792, 24.692, 24.592, 24.492$ and 24.392 , respectively, were available.

For the various values of μ_c/μ_e assumed, a series of parallel curves, determined by the right-hand sides of the equations (2), were obtained in the uv -plane; and it was found by

¹ *Ap. J.*, 96, 161, 1942.

² *An Introduction to the Study of Stellar Structure*, p. 352, Chicago, 1939.

³ *M.N.*, 92, 645, 1931-32.

interpolation that point-source envelopes could be fitted to convective cores for the values of $\log \kappa_0$ given in Table 1.

In Table 2 the quantities

$$q = \frac{r_i}{R} \quad \text{and} \quad \nu = \frac{M(r_i)}{M},$$

which give the fraction of the radius and the mass occupied by the convective core, are tabulated. The boundary of the configuration is reached at $\xi = \xi_1$, and $\psi = \psi_1$.

TABLE 1

μ_c/μ_e	$\log \kappa_0$	μ_c/μ_e	$\log \kappa_0$
1.0.....	24.737	1.4.....	24.929
1.1.....	24.817	1.6.....	24.961
1.2.....	24.871	1.8.....	24.982
1.3.....	24.905	2.0.....	24.992

TABLE 2

μ_c/μ_e	q	ν	ξ_1	ψ_1	$\frac{2}{5} \frac{\xi_1}{\psi_1} \mu_c$	$\frac{L_0}{\left[\frac{2}{5} \frac{\xi_1}{\psi_1} \mu_c\right]^{1/2}} \times 10^{-28}$
1.0	0.171	0.150	7.06	3.15	0.897	0.576
1.1	.152	.137	8.07	3.62	0.981	.662
1.2	.138	.126	9.06	4.10	1.060	.722
1.3	.126	.114	10.10	4.73	1.110	.763
1.4	.114	.101	11.35	5.56	1.144	.794
1.6	.100	.089	13.40	6.82	1.258	.815
1.8	.091	.083	15.12	7.88	1.382	.816
2.0	0.087	0.081	16.2	8.74	1.483	0.806

Assuming that the central temperature remains constant during the evolution, we can use the following relation to determine the effect of increasing μ_c/μ_e on the radius of the star:

$$T_c = \frac{2}{5} \frac{\xi_1}{\psi_1} \frac{\mu_c H}{k} \frac{GM}{R} \tag{3}$$

or

$$R = \frac{2}{5} \mu_c \frac{\xi_1}{\psi_1} \frac{H}{k} \frac{GM}{T_c} \tag{4}$$

The variation in R will thus be governed only by the factor $2\mu_c\xi_1/5\psi_1$ if T_c is assumed to remain constant. This factor is tabulated in Table 2.

For a homologous family of stellar configurations based on Kramers' law of opacity, the luminosity is given by a formula of the form

$$L = L_0 \frac{M^{5.5}}{\kappa_0 R^{0.5}} \mu_e^{7.5}, \tag{5}$$

where L_0 is a characteristic constant of the family which can be obtained directly from the integrations. If L , M , and R are expressed in solar units in the foregoing equation, then

$$L_0 = (\kappa_0)_{\text{int}}. \tag{6}$$

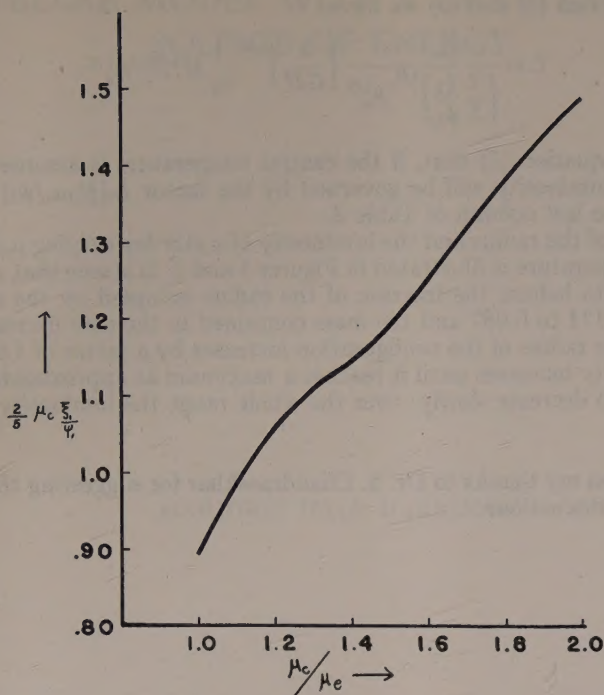


FIG. 1.—The curve illustrates the variation in the radius for constant central temperature as a function of the ratio of the mean molecular weight in the core to the mean molecular weight in the envelope. The ordinates represent the radius in certain units appropriate for the problem.

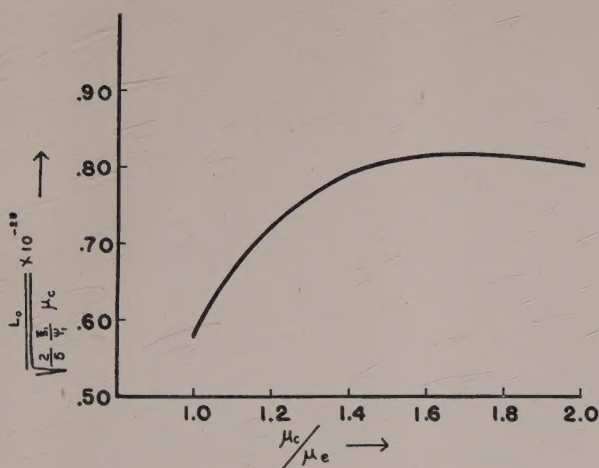


FIG. 2.—The curve illustrates the variation in the luminosity for constant central temperature as a function of the ratio of the mean molecular weight in the core to the mean molecular weight in the envelope. The ordinates represent the luminosity in certain units appropriate for the problem.

Combining equations (5) and (6) we have,

$$L = \frac{L_0}{\left[\frac{2}{5} \frac{\xi_1}{\psi_1}\right]^{1/2} \mu_c^{1/2}} \left[\frac{k}{GH}\right]^{0.5} \frac{1}{\kappa_0} M^5 T_c^{0.5} \mu_e^{7.5}. \quad (7)$$

It is clear from equation (7) that, if the central temperature is assumed constant, the variation in the luminosity will be governed by the factor $L_0[\frac{2}{5}\xi_1\mu_c/\psi_1]^{-1/2}$; this factor is tabulated in the last column of Table 2.

The variation of the radius and the luminosity of a star for varying μ_c/μ_e and for constant central temperature is illustrated in Figures 1 and 2. It is seen that as the hydrogen is transformed into helium the fraction of the radius occupied by the convective core decreases from 0.171 to 0.087 and the mass contained in the core decreases from 0.150 to 0.081, while the radius of the configuration increases by a factor of 1.65. At the same time the luminosity increases until it reaches a maximum at approximately $\mu_c/\mu_e = 1.8$ and then starts to decrease slowly; over the whole range the luminosity increases by a factor of 1.40.

I wish to express my thanks to Dr. S. Chandrasekhar for suggesting this problem and for many helpful discussions.

A STELLAR MODEL WITH A GRAVITATIONAL SOURCE OF ENERGY

MARJORIE HALL HARRISON

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A STELLAR MODEL WITH A GRAVITATIONAL SOURCE OF ENERGY

MARJORIE HALL HARRISON

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Received June 8, 1945

ABSTRACT

A stellar model consisting of a convective core and an envelope in which the energy generation is of the form $\epsilon \propto T$ is considered. It is found that the convective core extends to a fraction of 0.14 of the radius of the star and incloses 10.5 per cent of the mass. Further, the constants of proportionality in the equations for ρ_c , T_c , P_c , and L differ only slightly from those for the Cowling point-source model.

1. *Introduction.*—In this paper we propose to consider a stellar model with negligible radiation pressure, in which the law of energy generation has the form

$$\epsilon = \epsilon_0 T, \quad (1)$$

where ϵ denotes the energy liberated per gram of the stellar material at temperature T and where ϵ_0 is a constant. The study of such a model has gained renewed interest for stellar structure since it became apparent that the liberation of gravitational energy in consequence of a slow secular contraction of the star must play an important role in discussions relating to stellar evolution.¹ For under circumstances in which a star contracts homologously,² the rate of energy liberation is given by

$$\epsilon = -\frac{3\gamma - 4}{\gamma - 1} \frac{1}{R} \frac{dR}{dt} \frac{k}{\mu H} T, \quad (2)$$

where R denotes the radius of the star and the rest of the symbols have their usual meanings.

We should, however, point out that the study of stellar models based on the law (1) or (2) is not new. Indeed, since A. S. Eddington's³ discussion of the validity of his model " $\kappa\eta = \text{constant}$," it has been generally believed that the standard model on which stars are polytropes of index 3 describes the model $\epsilon \propto T$ very closely. Moreover, Biermann made an exact integration of the equations of equilibrium for a special case, making due allowance for the radiation pressure. But it does not seem that an integration of the important case in which the radiation pressure is neglected in the equation of hydrostatic equilibrium exists. It is the object of this paper to provide such an integration.

2. *The equations of the model.*—In our discussion of the model based on the law (1) we shall suppose that, in the parts of the star where the radiative gradient obtains, the law of opacity is the conventional law of Kramers:

$$\kappa = \kappa_0 \rho T^{-3.5}. \quad (3)$$

With this law the standard equations of equilibrium

$$\frac{k}{\mu H} \frac{d(\rho T)}{dr} = -\frac{GM(r)}{r^2} \rho; \quad \frac{dM(r)}{dr} = 4\pi \rho r^2, \quad (4)$$

¹ Cf., e.g., M. Schönberg and S. Chandrasekhar, *Ap. J.*, **96**, 161, 1942.

² For a discussion of the conditions under which such a contraction can take place see L. H. Thomas, *M.N.*, **91**, 122 and 619, 1930.

³ *The Internal Constitution of the Stars*, pp. 114 ff., Cambridge, England, 1926.

and

$$\frac{d p_r}{d r} = -\frac{\kappa_0 L(r)}{4 \pi c r^2} \rho^2 T^{-3.5}; \quad \frac{d L(r)}{d r} = 4 \pi \rho r^2 \epsilon_0 T, \quad (5)$$

valid everywhere except in the convective core at the center, can be reduced to the non-dimensional forms

$$\frac{d(\sigma \theta)}{d \xi} = -\frac{\psi \sigma}{\xi^2}; \quad \frac{d \psi}{d \xi} = \xi^2 \sigma, \quad (6)$$

and

$$\frac{d \theta}{d \xi} = -\frac{Q \lambda \sigma^2}{\xi^2 \theta^{6.5}}; \quad \frac{d \lambda}{d \xi} = \theta \xi^2 \sigma, \quad (7)$$

by the transformation

$$r = R \xi; \quad \rho = \rho_0 \sigma; \quad T = T_0 \theta; \quad M(r) = M \psi; \quad L(r) = L_0 \lambda, \quad (8)$$

with

$$\rho_0 = \frac{M}{4 \pi R^3}; \quad T_0 = \frac{\mu H}{k} \frac{GM}{R} \quad (9)$$

and

$$Q = \kappa_0 \epsilon_0 \frac{3 R^2}{4 a c} \frac{\rho_0^3}{T_0^{6.5}} = \kappa_0 L_0 \frac{3}{16 \pi a c R} \frac{\rho_0^2}{T_0^{7.5}}. \quad (10)$$

It is found that, as we approach the center of the configuration along a solution of the equations (6) and (7), the radiative gradient becomes unstable at a determinate point, where the effective polytropic index becomes $\frac{3}{2}$. At this point the convective equations must accordingly replace the radiative, and

$$\sigma = \theta^{3/2}; \quad (11)$$

and the equations (6) can be reduced to the Lane-Emden equation of index $\frac{3}{2}$.

3. *The construction of the model.*—Near $\xi = 1$ the equations (6) and (7) can be approximately integrated. For in the immediate neighborhood of the boundary of the star, $M(r)$ and $L(r)$ remain constant to a high degree of accuracy. Accordingly, letting ψ and λ have their respective boundary values, which we shall denote by ψ_1 and λ_1 , the first of the two equations in (6) and (7) can be re-written in the form

$$\frac{d(\sigma \theta)}{d(\theta^4)} = \frac{\psi_1}{\lambda_1} \frac{1}{4 Q} \frac{\theta^{3.5}}{\sigma} \quad (12)$$

and

$$\frac{d(\theta^4)}{d \xi} = -\frac{4 Q \lambda_1 \sigma^2}{\xi^2 \theta^{3.5}}. \quad (13)$$

Now let

$$\sigma = \frac{\psi_1^{15}}{2 Q^8 \lambda_1^{17/4}} S; \quad \theta = \frac{1}{Q^2} \frac{\psi_1^4}{\lambda_1} t \quad (14)$$

and

$$K = \frac{\psi_1^3}{Q^2 \lambda_1}. \quad (15)$$

Introducing these substitutions into equations (12) and (13) and making the further substitutions

$$t = \frac{u^2}{K^2}; \quad S = z \frac{u^6}{K^7}, \quad (16)$$

we have

$$\frac{1}{8} z u \frac{dz}{du} = u - z^2 \quad (17)$$

and

$$\frac{du}{d\xi} = -\frac{K}{8} \frac{z^2}{\xi^2 u^2} \quad (18)$$

The solutions of equations (17) and (18) satisfying the boundary condition that z and u tend to zero simultaneously are

$$z^2 = \frac{1}{17} u; \quad \frac{1}{\xi} = 1 + \frac{17}{4} \frac{u^2}{K} \quad (19)$$

The corresponding solutions for σ and θ are

$$\sigma = \frac{\lambda_1^{1/4}}{\psi_1^6} \frac{Q^6}{2} z \omega^{12}; \quad \theta = \frac{\lambda_1}{\psi_1^2} Q^2 \omega^4. \quad (20)$$

For any specified value of Q the foregoing equations will enable us to derive σ , θ , and the various other quantities for some value of $\xi < 1$. In practice it is found that these equations provide sufficient accuracy up to $\xi = 0.92$.

For values of $\xi < 0.92$ the solutions thus started will have to be continued by numerical methods. For this purpose it is convenient to introduce the variable

$$x = \frac{1}{\xi} \quad (21)$$

and to re-write equations (6) and (7) in the forms

$$\frac{d(\sigma\theta)}{dx} = \psi\sigma; \quad \frac{d\psi}{dx} = -\frac{\sigma}{x^4} \quad (22)$$

and

$$\frac{d\theta}{dx} = \frac{Q\lambda\sigma^2}{\theta^{6.5}}; \quad \frac{d\lambda}{dx} = -\frac{\theta\sigma}{x^4} \quad (23)$$

Owing to the nature of the differential equations to be solved, it is found more convenient to do the integration in terms of logarithmic variables. Using an asterisk (*) to denote the logarithm to the base 10, we have

$$\sigma^* = \log \sigma; \quad \theta^* = \log \theta; \quad \psi^* = \log \psi; \quad x^* = \log x; \quad \lambda^* = \log \lambda. \quad (24)$$

Let

$$\varphi = \sigma\theta; \quad (25)$$

then

$$\varphi^* = \log \varphi = \log \sigma + \log \theta. \quad (26)$$

The equations to be integrated then become

$$\left. \begin{aligned} \frac{d\varphi^*}{dx^*} &= \frac{x}{\varphi} \psi\sigma = 10^{\psi^* + \sigma^* + x^* - \varphi^*}, \\ \frac{d\psi^*}{dx^*} &= -\frac{x}{\psi} \frac{\sigma}{x^4} = -10^{-3x^* + \sigma^* - \psi^*}, \\ \frac{d\theta^*}{dx^*} &= \frac{x}{\theta^{7.5}} Q\lambda\sigma^2 = Q 10^{x^* + \lambda^* + 2\sigma^* - 7.5\theta^*}, \\ \frac{d\lambda^*}{dx^*} &= -\frac{\theta\sigma}{\lambda x^3} = -10^{\theta^* + \sigma^* - \lambda^* - 3x^*}. \end{aligned} \right\} \quad (27)$$

It is thus seen that for any specified Q the equations can be integrated, starting from the boundary and going inward. However, as we go along such a solution, a point is generally reached where the radiative gradient breaks down. This point, ξ_i , where the convective gradient sets in is defined by the place where the effective polytropic index

$$n = \frac{1}{Q} \frac{\theta^{6.5} \psi}{\lambda \sigma^2} - 1 \quad (28)$$

TABLE 1
MASS, LUMINOSITY, TEMPERATURE, AND DENSITY
DISTRIBUTIONS FOR THE MODEL $\epsilon \propto T$

ξ	ψ	λ	θ	σ
1.0000	1.0000	1.0000		
0.9701	1.0000	1.0000	0.0073	
0.9410	1.0000	1.0000	.0147	
0.9175	1.0000	1.0000	.0212	
0.8810	0.9999	1.0000	.0318	0.0063
0.8504	0.9996	1.0000	.0414	0.0145
0.8207	0.9993	1.0000	.0514	0.0272
0.7881	0.9984	1.0000	.0632	0.0538
0.7645	0.9975	0.9999	.0724	0.0842
0.7342	0.9956	0.9997	.0851	0.1410
0.7052	0.9929	0.9994	.0983	0.2258
0.6770	0.9891	0.9991	.1120	0.3455
0.6371	0.9811	0.9981	.1336	0.6093
0.6118	0.9739	0.9971	.1486	0.8594
0.5756	0.9600	0.9948	.1721	1.3783
0.5528	0.9484	0.9927	.1884	1.8424
0.5202	0.9271	0.9885	.2137	2.7585
0.4896	0.9011	0.9825	.2401	3.9917
0.4514	0.8588	0.9716	.2767	6.2392
0.4163	0.8026	0.9566	.3146	9.3058
0.3839	0.7504	0.9373	.3535	13.3010
0.3540	0.6869	0.9135	.3929	18.2850
0.3264	0.6196	0.8858	.4317	24.2826
0.2891	0.5165	0.8382	.4903	34.8168
0.2560	0.4172	0.7868	.5460	46.7441
0.2267	0.3274	0.7354	.5984	59.1696
0.2007	0.2504	0.6875	.6469	71.2087
0.1707	0.1694	0.6329	.7054	85.4250
0.1419	0.1050	0.5854	.7642	98.0942
0.1119	0.0551	0.1966	.8214	109.4
0.0819	0.0227	0.0430	.8682	118.9
0.0519	0.0060	0.0036	.9017	125.9
0.0219	0.0006		.9206	129.8
			0.9251	130.8

becomes $\frac{3}{2}$. The solution interior to ξ_i must be described by a Lane-Emden function $\theta_{3/2}$. For an initially arbitrary choice Q , the solution obtained by the integration cannot be fitted to $\theta_{3/2}$. The condition that the fit be made determines Q . Now the conditions at the interface are that the values of the temperature, pressure, and mass of the core should be identical at the interface, along with the condition that the effective polytropic index n attain the value 1.5 at the interface. We then have

$$P(r_i)_c = P(r_i)_e; \quad T(r_i)_c = T(r_i)_e; \quad M(r_i)_c = M(r_i)_e; \quad n = 1.5, \quad (29)$$

where P , M , and T denote the total pressure, the mass within the radius r , and the temperature, respectively, and where c refers to the core, e to the envelope, and i to the interface.

Introducing the homology invariant quantities u and v , defined as in Chandrasekhar's *Stellar Structure* (pp. 146 ff.), we have for the conditions at the interface:

$$u_i = \frac{4\pi\rho_e(r_i)r_i^3}{M_e(r_i)} = u_e = \left(\frac{\sigma}{\psi} \frac{1}{x^3}\right)_{r=r_i} \quad (30)$$

and

$$v_i = \frac{2}{5} \frac{GM_e(r_i)}{r_i T_e(r_i)} \frac{\mu H}{k} = v_e = \left(\frac{2}{5} \frac{\psi}{\theta} x\right)_{r=r_i} \quad (31)$$

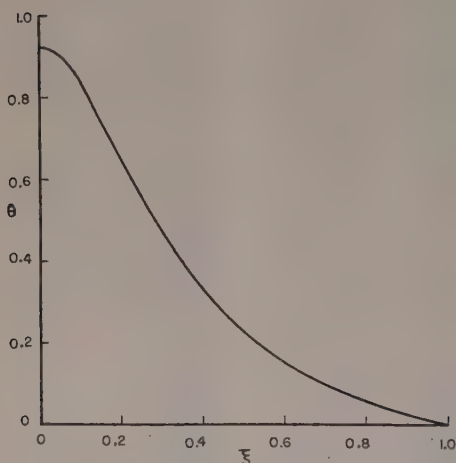


FIG. 1.—The temperature distribution in the model with a gravitational source of energy.

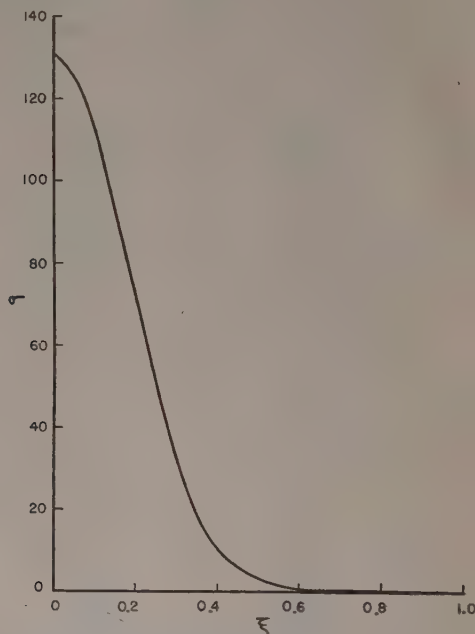


FIG. 2.—The density distribution in the model with a gravitational source of energy.

In terms of logarithms these conditions become

$$\text{and} \quad \left. \begin{aligned} \log u_e &= \varphi^* - \theta^* - 3x^* - \psi^*; \\ \log v_e &= -0.397940 + \psi^* + x^* - \theta^*. \end{aligned} \right\} \quad (32)$$

The envelope equations were integrated for values of $\log Q = -5.7265, -5.85, -5.9, -6, \text{ and } -7$. All calculations were carried out logarithmically to six decimal places. When the point was reached at which the radiative gradient broke down, u_e and v_e were calculated. For the regions of the star inside $r = r_i$ the Fairclough⁴ solutions of the Lane-Emden equation of index $\frac{3}{2}$ were used.

4. *Results.*—From the integrations we find that

$$\left. \begin{aligned} \log Q &= -5.8864; & u_e &= 2.6647; & v_e &= 0.3885, \\ \xi_i &= 0.1419; & \psi_i &= 0.1050; & \lambda_i &= 0.5854, \\ \theta_i &= 0.7642; & \sigma_i &= 98.0942; \end{aligned} \right\} \quad (33)$$

and from equation (10)

$$\log \kappa_0 = 24.8379. \quad (33')$$

⁴ *M.N.*, 92, 645, 1931-1932.

By interpolating in the Lane-Emden tables we find that $\theta_i^{3/2} = 0.7501$. This determines the central value of σ . For, we must have⁵

$$\sigma_i = \sigma_c \theta_i^{3/2}. \quad (34)$$

From equation (34) and the known value of $\theta_i^{3/2} (= 0.7501)$

$$\sigma_c = 130.8. \quad (35)$$

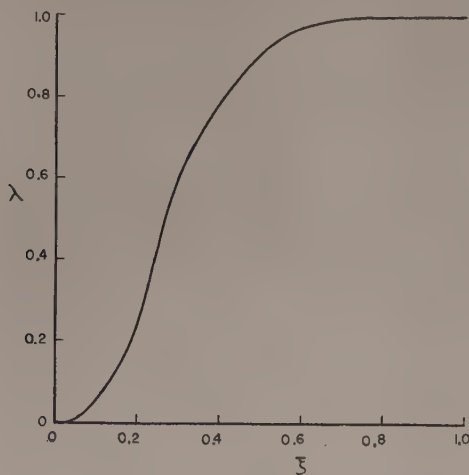


FIG. 3.—The luminosity distribution in the model with a gravitational source of energy

Also, since

$$\bar{\rho} = \frac{M}{\frac{4}{3}\pi R^3} = 3\rho_0 \quad (36)$$

and

$$\rho_c = \rho_0 \sigma_c, \quad (37)$$

we find

$$\rho_c = 43.6\bar{\rho}. \quad (38)$$

Moreover,

$$\sigma_i \theta_i = K \theta_c^{5/3} \theta_i^{5/2} \quad (39)$$

or

$$K = 0.0359; \quad (40)$$

and

$$\sigma_c \theta_c = K \sigma_c^{5/3}, \quad (41)$$

or

$$\theta_c = 0.9251. \quad (42)$$

Finally, we have

$$\rho_c = 43.6\bar{\rho},$$

$$T_c = T_0 \theta_c = 0.9251 \frac{\mu H}{k} \frac{GM}{R},$$

$$P_c = \frac{k}{\mu H} \rho_c T_c = 9.625 \frac{GM^2}{R^4}, \quad (43)$$

$$L = 6.885 \times 10^{24} \frac{1}{\kappa_0} \frac{M^{5.5}}{R^{0.5}} \mu^{7.5},$$

where L , M , and R are expressed in the corresponding solar units.

⁵ S. Chandrasekhar, *An Introduction to the Study of Stellar Structure*, p. 87, Chicago: University of Chicago Press, 1939.

5. *Conclusions.*—It is found that the convective core extends to a fraction 0.14 of the radius of the star and incloses 10.5 per cent of the mass. The march of ψ , λ , θ , and σ for this model is shown in Table 1 (see also Figs. 1, 2, and 3). Further, we notice that the constants of proportionality in equations (43) differ only slightly from those for the Cowling point-source model.

I wish to express my thanks to Dr. S. Chandrasekhar for suggesting this problem and for many helpful discussions.

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ABSTRACT

Stellar models, consisting of partially degenerate isothermal cores and point-source envelopes, are considered in which the ratio of the mean molecular weights in the core and the envelope takes two values, namely, 1 and 2. The results presented in this paper do not agree with those of a similar calculation attempted by Gamow and Keller and do not support their suggestion that allowing for the degeneracy of the isothermal core devoid of hydrogen will significantly alter the early course of evolution of the main-sequence stars.

1. *Introduction.*—In their discussion of the evolution of the main-sequence stars M. Schönberg and S. Chandrasekhar¹ conclude, on the basis of their study of stellar models with isothermal cores, that there is an upper limit to the fraction of the mass which can exist in the form of isothermal cores devoid of hydrogen. On the strength of this result they say that “it is difficult to escape the conclusion that beyond this point [where the core devoid of hydrogen has increased to its maximum size] the star must evolve through nonequilibrium configurations.” But it has recently been claimed by G. Gamow² that this is a “paradoxical result” which arises only because of the “arbitrary assumption” made by Schönberg and Chandrasekhar that the isothermal core is gaseous and that, if once this assumption is dropped to allow for the possible degeneracy of the core, the isothermal regions can spread through the whole star. Indeed, G. Gamow and G. Keller³ believe that they have shown that stellar models consisting of partially degenerate isothermal cores and radiative envelopes can be constructed which will account for the giants and their energy production in terms of the carbon cycle. These conclusions are at such variance with those of Schönberg and Chandrasekhar that we have re-examined the problem *de novo* and find that we cannot substantiate the claims of either Gamow or of Gamow and Keller. But before we enter into the details of the construction of the models, it is of interest to examine the problem in a general way and see why it is that allowing for the possible degeneracy of the core cannot affect in any significant way the conclusions of Schönberg and Chandrasekhar.⁴

According to the calculations of Schönberg and Chandrasekhar, the central temperature and the density of the configuration which incloses the maximum fraction (~ 10 per cent) of the mass in the isothermal core (assumed gaseous) are given by

$$T_c = 0.713 \frac{\mu_c H}{k} \frac{GM}{R} \quad (1)$$

and

$$\rho_c = 3210 \frac{3M}{4\pi R^3}, \quad (2)$$

¹ *A. J.*, 96, 161, 1942.

² *Phys. Rev.*, 67, 120, 1945.

³ *Rev. Mod. Phys.*, 17, 125, 1945.

⁴ I am indebted to Dr. S. Chandrasekhar for the discussion which follows.

where the various symbols have their usual meaning. It is now clear that the assumption of the gaseous nondegenerate nature of the core will be a valid one if the pressure computed according to the perfect gas formula

$$P_{\text{nondeg}} = \frac{k}{\mu_c H} \rho_c T_c \quad (3)$$

exceeds that given by the degenerate formula⁵

$$P_{\text{deg}} = \frac{K_1}{\mu_c^{5/3}} \rho_c^{5/3}, \quad (4)$$

where

$$K_1 = \frac{1}{20} \left(\frac{3}{\pi} \right)^{2/3} \frac{h^2}{m_e H^{5/3}} = 9.91 \times 10^{12} \quad (5)$$

is the unrelativistic degenerate constant. (It will be noted that we are ignoring the relativistic modification of the formula giving the pressure of a degenerate case. This is not of importance in the present context. But it is of importance in another connection, to which we shall presently make reference.)

The validity of the assumption concerning the nondegeneracy of the core requires, then, that

$$\frac{k}{\mu_c H} \rho_c T_c > \frac{K_1}{\mu_c^{5/3}} \rho_c^{5/3}. \quad (6)$$

Substituting for ρ_c and T_c from equations (1) and (2), the foregoing inequality reduces to

$$\left(\frac{M}{\odot} \right)^{1/3} \frac{R}{R_{\odot}} > \frac{2.00}{\mu_c^{5/3}}. \quad (7)$$

On the other hand, if a temperature of 20,000,000 K° is accepted as a criterion for the operation of the carbon cycle, we should have

$$2 \times 10^7 = 0.713 \frac{\mu_c H}{k} \frac{GM}{R}. \quad (8)$$

Eliminating R between the relations (7) and (8), we find

$$\frac{M}{\odot} > \frac{1.95}{\mu_c^2} = 0.5 \quad (\mu_c = 2). \quad (9)$$

In other words, for $T_c = 2.0 \times 10^7$ degrees (K), degeneracy cannot be a significant factor in the evolution of stars of masses greater than $\frac{1}{2} \odot$ during the stage of the burning of the hydrogen. For stars of small mass ($M < 0.5 \odot$), incipient degeneracy will occur already during the early stages. This is, of course, well known from the study of the M dwarfs themselves and is not of any special relevance in our present context.

Turning next to a somewhat different aspect of the same problem,⁶ we first recall that the maximum mass in hydrostatic equilibrium which can be degenerate is $6.6 \odot \mu_e^{-2}$. Under no circumstances can degeneracy spread over a larger mass. For $\mu_e \sim \mu_c = 2$ this is $1.65 \odot$; and, accordingly, for stars of large mass ($M > 3 - 4 \odot$), degeneracy can never spread over more than a fraction of its mass. This, again, is a well-known result; but it should be emphasized in this connection that, in order that we may have $1.65 \odot$ in the

⁵ S. Chandrasekhar, *An Introduction to the Study of Stellar Structure*, p. 421, Chicago: University of Chicago Press, 1939.

⁶ Chandrasekhar, *op. cit.*, p. 438. Here μ_e is the mean molecular weight appropriate for computing the pressure of the electron gas. Since we are dealing with situations in which no hydrogen is present, we need not distinguish between μ_c and μ_e .

degenerate core, the density throughout the core must (normally!) exceed 10^8 gm/cm^3 . Such configurations must be, accordingly, of white-dwarf dimensions and can therefore have no bearing on the structure and constitution of the giant stars.

When the two lines of arguments presented in the preceding paragraphs are combined, it is evident that degeneracy as a factor in the evolution of the main-sequence stars with masses exceeding $0.75\odot$ can play no significant role during the early stages, when hydrogen is being burned. This situation is slightly different for stars of small masses ($M < 0.5\odot$). But, as we have already pointed out, these stars are already incipiently degenerate on the main sequence. Evolution of these stars is, therefore, principally one along a sequence in which the transition is directly to the white-dwarf stages. Since degeneracy is an important factor, temperature is irrelevant to the situation, and the isothermal character of the core devoid of hydrogen plays a minor role. It is thus apparent that, where the isothermal character of the core is significant, degeneracy ceases to play a role. There is thus a mutual exclusiveness of the two factors.

In spite of what we have said in the preceding paragraphs, it is definitely of interest to study stellar models in which we make allowance for the possible degeneracy of the core. These models are studied in this paper. But we may draw attention, even at this stage, to the fact that our results are in disagreement with those of Gamow and Keller. We are unable to follow their method of fitting in detail, but we feel that there can be no doubt about the correctness of our results.⁷

2. *The equations of the models.*—In the absence of relativity effects the Fermi-Dirac equation of state of the partially degenerate isothermal core with negligible radiation pressure can be expressed parametrically in the form

$$\rho = \frac{4\pi\mu_e H (2mkT)^{3/2}}{h^3} F_{1/2}(\phi), \quad (10)$$

and

$$P = \frac{4\pi (2m)^{3/2} (kT)^{5/2}}{h^3} \mathfrak{F}_{3/2}(\phi), \quad (11)$$

where m is the mass of the electron; h , the Planck constant; H , the mass of the proton; k , the Boltzmann gas constant; μ_e , the molecular weight of an electron gas; and $F_{1/2}(\phi)$, the Fermi-Dirac⁸ functions defined by

$$F_\nu(\phi) = \int_0^\infty \frac{u^\nu du}{e^{-\phi+u} + 1} \quad (12)$$

and

$$\frac{2}{3} F_{3/2}(\phi) = \mathfrak{F}_{3/2}(\phi), \quad (13)$$

where ϕ is a measure of the degree of degeneracy.

With the foregoing equation of state the equation of equilibrium

$$\frac{1}{r^2} \frac{d}{dr} \left(\frac{r^2}{\rho} \frac{dP}{dr} \right) = -4\pi G \rho \quad (14)$$

can be reduced to the form

$$\frac{1}{\eta^2} \frac{d}{d\eta} \left(\eta^2 \frac{d\phi}{d\eta} \right) = -F_{1/2}(\phi), \quad (15)$$

where

$$r = a_1 \eta, \quad \text{and} \quad a_1 = \left[\frac{h^3}{G (4\pi\mu_e H)^2 (2m)^{3/2} (kT)^{1/2}} \right]^{1/2}. \quad (16)$$

⁷ One error to which we may draw attention is that the formula (18) quoted by Gamow and Keller for the pressure of a degenerate gas is incorrect by a factor 8, but this may be an error in the type.

⁸ J. McDougall and E. C. Stoner, *Phil. Trans., R.S., A*, 237, 67, 1938.

The mass of the material inclosed up to a radius η is given by

$$M(\eta) = 4\pi \int_0^\eta \rho r^2 dr = 4\pi \alpha_1^3 \int_0^\eta \rho \eta^2 d\eta. \quad (17)$$

In order to express ρ in terms of ϕ and η we make use of equations (10) and (15); thus

$$\rho = -\frac{4\pi\mu_e H (2mkT)^{3/2}}{h^3} \frac{1}{\eta^2} \frac{d}{d\eta} \left(\eta^2 \frac{d\phi}{d\eta} \right). \quad (18)$$

Hence the required mass relation is

$$M(\eta) = -\frac{(4\pi)^2 \alpha_1^3 H \mu_e (2mkT)^{3/2}}{h^3} \eta^2 \frac{d\phi}{d\eta}. \quad (19)$$

The standard equations of equilibrium for the point-source envelope,

$$\frac{k}{\mu H} \frac{d}{dr} (\rho T) = -\frac{GM(r)\rho}{r^2}; \quad \frac{dM(r)}{dr} = 4\pi \rho r^2, \quad (20)$$

and

$$\frac{a}{3} \frac{d}{dr} (T^4) = -\frac{\kappa_0 L}{4\pi c r^2} \frac{\rho^2}{T^{3.5}}, \quad (21)$$

where the law of opacity is the conventional law of Kramers,

$$\kappa = \kappa_0 \rho T^{-3.5}, \quad (22)$$

can be reduced to the nondimensional forms

$$\frac{d}{d\xi} (\sigma \theta) = -\frac{5}{2} \frac{\sigma \psi}{\xi^2}; \quad \frac{d\psi}{d\xi} = \sigma \xi^2, \quad (23)$$

$$\frac{d\theta}{d\xi} = \frac{-Q\sigma^2}{\xi^2 \theta^{6.5}} \quad (24)$$

by the transformation

$$r = R\xi; \quad \rho = \rho_0 \sigma; \quad T = T_0 \theta; \quad M(r) = M\psi, \quad (25)$$

with

$$R^3 \rho_0 = \frac{M}{4\pi}; \quad RT_0 = \frac{2}{5} \frac{\mu H}{k} GM \quad (26)$$

and

$$Q = \frac{\kappa_0 L}{16\pi} \frac{3}{ac} \frac{\rho_0^2}{RT_0^{7.5}}. \quad (27)$$

According to equations (20), (23), and (25), the mass of the material inclosed up to a radius r_1 is given by

$$M(r_1) = -\frac{1}{G} \frac{r_1^2}{\rho} \frac{dP}{dr} = -\frac{2}{5} M \frac{\xi^2}{\sigma} \frac{d(\sigma \theta)}{d\xi}. \quad (28)$$

3. *The equations of fit.*—Let the interface occur at $\xi = \xi_i$ and $\eta = \eta_i$. We then have two sets of equations for the absolute value of the radius, mass, and pressure at the interface. Our equations of fit are derived from the requirement that the two sets agree. In this manner we find

$$-\frac{2}{5} M \frac{\xi^2}{\sigma} \frac{d(\sigma \theta)}{d\xi} = -\frac{(4\pi)^2 \alpha_1^3 H \mu_e (2mkT)^{3/2}}{h^3} \eta^2 \frac{d\phi}{d\eta}, \quad (29)$$

$$R\xi = \left[\frac{h^3}{G(4\pi\mu_e H)^2 (2m)^{3/2} (kT)^{1/2}} \right]^{1/2} \eta, \quad (30)$$

$$\rho_0 \sigma = \frac{4\pi\mu_e H (2mkT)^{3/2}}{h^3} F_{1/2}(\phi) x, \quad (31)$$

and

$$\frac{k}{\mu H} \rho_0 T_0 \sigma \theta = \frac{4\pi (2m)^{3/2} (kT)^{5/2}}{h^3} \frac{2}{3} F_{3/2}(\phi) \quad (32)$$

where, in equation (31)

$$x = \frac{\rho_{\text{env}}}{\rho_{\text{core}}} \sim \frac{\mu_{\text{env}}}{\mu_{\text{core}}} \quad (33)$$

Equation (33) allows for a discontinuity of the density at $\xi = \xi_i$ to correspond with the difference in the mean molecular weights of the core and the envelope.

This system of equations can be reduced to one involving only two homology invariant combinations of σ , θ , and their derivatives σ' , θ' , and ξ . Raise equation (30) to the third power, divide by (29) and multiply by 4π times (31). We are left with

$$-\frac{5}{2} \frac{\sigma^2 \xi}{(\sigma \theta)'} = -\frac{\eta}{\phi} F_{1/2}(\phi) x, \quad (34)$$

which we re-write in the form

$$-\frac{5}{2} \frac{\sigma^2 \xi}{(\sigma \theta)'} = u(\xi) = U(\eta) = -\frac{\eta}{\phi} F_{1/2}(\phi) x. \quad (35)$$

Divide (29) by (32), multiply by (31), and divide by (30). We then have

$$-\frac{(\sigma \theta)'}{\sigma \theta} \xi = -\frac{F_{1/2}(\phi)}{\frac{2}{3} F_{3/2}(\phi)} \eta \phi' x \quad (36)$$

which we re-write in the form

$$-\frac{2}{5} \frac{(\sigma \theta)'}{\sigma \theta} \xi = v(\xi) = V(\eta) = -\frac{2}{5} \frac{F_{1/2}(\phi)}{\frac{2}{3} F_{3/2}(\phi)} \eta \phi' x. \quad (37)$$

4. *The construction of the models.*—Our fitting conditions are those which are usually used, namely, that the mass, radius, and pressure are continuous at the interface; and we have also made allowance for the difference in the mean molecular weights of the core and envelope by making the density discontinuous at the interface (cf. eq. [33]). However, it should be pointed out in this connection that it is not strictly correct to use the perfect gas equation for $\xi \geq \xi_i$ and change over abruptly to the Fermi-Dirac equation of state interior to ξ_i . But this cannot be avoided if we want to use the available integrations of the point-source envelope. It seems reasonable to suppose that for each value of the degree of degeneracy there would be a finite number of such models possible. With the available integrations of the point-source envelope we found that this supposition was true and that for any degenerate core and point-source envelope there were, at most, two such models. For each value of the degree of degeneracy a certain mass was obtained; and in order to determine the values of the physical characteristics for any desired mass, we must interpolate across these curves.

In order to determine the physical characteristics of any composite configuration, it is necessary to determine if there are any solutions which, besides satisfying the equations describing the core and the envelope, also satisfy the interfacial conditions. The procedure in such an investigation is to make use of the homology invariant quantities u and v^9 and transfer our curves to the u, v plane. In addition to the homology invariant quantities u and v which describe the point-source envelope, we introduce analogous functions U and V to describe the isothermal core. In this manner we are able to transfer the available isothermal and point-source solutions to a common plane. Each intersection between the two sets of curves then represents a possible composite configuration of the

⁹ Chandrasekhar, *op. cit.*, pp. 146 ff.

type desired. Obviously, we cannot choose the mass of the core as an independent variable but must determine it as a result of the fitting conditions and the equations describing the core and envelope.¹⁰

For the fitting we had two sets of integrations of the point-source envelope available. One set of these integrations is due to Miss I. Nielsen and is for values of $\log Q = \bar{3}.2494, \bar{3}.1494, \bar{3}.0494, \bar{4}.9494, \bar{4}.8494, \bar{4}.7494$, and $\bar{4}.6494$; the other set is due to M. Schönberg and is for values of $\log Q = \bar{3}.7894, \bar{3}.1894, \bar{3}.2894, \bar{3}.3894, \bar{3}.4394, \bar{3}.4894, \bar{3}.5894$, and $\bar{3}.9394$.¹¹ The integrations by G. Wares¹² for $\phi_0 = 0, 2, 3, 5, 10$, and 20 were used for the isothermal core.

A graphical method was used to determine the homology invariant functions u and v for each intersection between the point-source envelopes and isothermal cores; the values for ξ_i, ψ_i , and η_i were then determined by interpolation in the tables.

5. *The physical characteristics of the models.*—The mass, radius, luminosity, and ratio of the central to the mean density can be readily derived from equations (10), (16), (19), (25), and (27). The formulae are given for $\kappa_0 = 4 \times 10^{25}$ and in terms of the sun's radius and mass, with T in units of 2×10^7 degrees K.

Radius of the core:

$$\frac{r}{R_\odot} \mu_e T^{0.25} = 0.02236 \eta_i. \quad (38)$$

Mass of the core:

$$\frac{M(\eta)}{M_\odot} \mu_e^2 T^{-0.75} = -1.9351 \eta_i^2 \phi_i' \times 10^{-2}. \quad (39)$$

Radius of the configuration:

$$\frac{R}{R_\odot} \mu_e T^{0.25} = \frac{0.02236 \eta_i}{\xi_i}. \quad (40)$$

Mass of the configuration:

$$\frac{M}{M_\odot} \mu_e^2 T^{-0.75} = -\frac{1.9351 \times 10^{-2} \eta_i^2 \phi_i'}{\psi_i}. \quad (41)$$

Luminosity of the configuration:

$$\frac{L}{L_\odot} \mu_e^3 T^{-4.25} = \frac{1}{\theta_i^{7.5}} \frac{\eta_i \sigma_i^2 Q \times 3.359 \times 10^{-4}}{\xi_i [F_{1/2}(\phi_i)]^2}. \quad (42)$$

Ratio of the central to the mean density:

$$\frac{\rho_c}{\bar{\rho}} = \frac{0.3336 \eta_i^3 \psi_i}{\xi_i^2 (-\eta_i^2 \phi_i')} F_{1/2}(\phi_0). \quad (43)$$

The quantities (40), (41), (42), and (43) are tabulated in Table 1 for $\mu_{\text{core}} | \mu_{\text{env}} = 1$ and in Table 2 for $\mu_{\text{core}} | \mu_{\text{env}} = 2$.

¹⁰ This is where our method differs from that used by G. Gamow and G. Keller.

¹¹ I am indebted to Dr. Schönberg for making these integrations available.

¹² "Partially Degenerate Stellar Models" (a dissertation submitted to the Faculty of the Division of the Physical Sciences in candidacy for the degree of Doctor of Philosophy, Department of Astronomy and Astrophysics, University of Chicago Libraries, 1940).

TABLE 1

PHYSICAL CHARACTERISTICS OF THE COMPOSITE MODEL FOR $x = 1$ AND VARIOUS VALUES OF ϕ_0

$Q \times 10^3$	ξ_i	ψ_i	$\mu_e T^{0.25} R/R_\odot$	$\mu_e^2 T^{-0.75} M/M_\odot$	$\mu_e^3 T^{-4.25} L/L_\odot$	$\rho_c/\bar{\rho}$
(a) $\phi_0 = 0$						
1.120.....	0.1466	0.1172	0.3974	0.4773	0.246×10^{-2}	51.5
1.410.....	.1678	.1987	0.4797	0.5831	$.942 \times 10^{-2}$	74.1
1.547.....	.1698	.2150	0.4993	0.5996	$.121 \times 10^{-1}$	81.3
1.776.....	.1742	.2532	0.5598	0.6662	$.243 \times 10^{-1}$	103
1.947.....	.1753	.2710	0.5929	0.6983	$.516 \times 10^{-1}$	117
2.451.....	.1764	.3112	0.6758	0.7642	$.728 \times 10^{-1}$	158
2.750.....	.1749	.3262	0.7224	0.7973	.103	185
3.086.....	.1718	.3417	0.7870	0.8412	.153	227
3.885.....	.1654	.3614	0.9160	0.9295	.320	324
6.157.....	.1551	.3950	1.169	1.058	.947	588
8.698.....	.1350	.4043	1.720	1.339	$.442 \times 10$	149×10
6.157.....	0.0875	0.2249*	17.41	10.77	0.989×10^5	192×10^3
(b) $\phi_0 = 2$						
1.120.....	0.1638	0.1530	0.2241	0.3370	0.132×10^{-3}	48.2
1.410.....	.1928	.2641	.2581	.3822	$.532 \times 10^{-3}$	65.0
1.547.....	.1974	.2893	.2662	.3884	$.714 \times 10^{-3}$	70.2
1.776.....	.2028	.3330	.2886	.4118	$.132 \times 10^{-2}$	84.4
1.947.....	.2050	.3532	.2995	.4215	$.175 \times 10^{-2}$	92.3
2.451.....	.2069	.3981	.3318	.4468	$.349 \times 10^{-2}$	118
2.750.....	.2060	.4152	.3488	.4585	$.471 \times 10^{-2}$	134
3.086.....	.2030	.4322	.3720	.4721	$.661 \times 10^{-2}$	157
3.885.....	.1992	.4578	.4122	.4971	$.116 \times 10^{-1}$	204
6.157.....	.1848	.4900	.5056	.5397	$.296 \times 10^{-1}$	346
8.698.....	0.1689	0.5013	0.6304	0.6015	0.728×10^{-1}	602
(c) $\phi_0 = 3$						
1.120.....	0.1763	0.1824	0.1884	0.3273	0.548×10^{-4}	46.9
1.410.....	.2131	.3216	.2117	.3582	$.250 \times 10^{-3}$	60.7
1.547.....	.2200	.3547	.2182	.3650	$.357 \times 10^{-3}$	65.3
1.776.....	.2256	.3994	.2327	.3784	$.630 \times 10^{-3}$	76.4
1.947.....	.2288	.4222	.2398	.3845	$.833 \times 10^{-3}$	82.3
2.451.....	.2312	.4685	.2614	.4007	$.161 \times 10^{-2}$	102
2.750.....	.2309	.4875	.2728	.4078	$.218 \times 10^{-2}$	114
3.086.....	.2267	.5010	.2888	.4172	$.288 \times 10^{-2}$	133
3.885.....	.2249	.5284	.3109	.4288	$.476 \times 10^{-2}$	161
6.157.....	.2112	.5619	.3723	.4577	$.118 \times 10^{-1}$	258
8.698.....	0.1976	0.5787	0.4316	0.4800	0.223×10^{-1}	384
(d) $\phi_0 = 5$						
1.120.....	0.2116	0.2758	0.1508	0.3481	0.232×10^{-4}	44.6
1.547.....	.2814	.5330	.1664	.3657	$.264 \times 10^{-4}$	57.1
1.776.....	0.2874	0.5760	0.1737	0.3706	0.452×10^{-4}	64.3

* This intersection corresponds to one for which ψ_i has already reached its maximum and thus is not a physically reliable solution.

TABLE 1—Continued

$Q \times 10^3$	ξ_i	ψ_i	$\mu_e T^{0.26} R/R_\odot$	$\mu_e^2 T^{-0.76} M/M_\odot$	$\mu_e^3 T^{-4.26} L/L_\odot$	$\rho c/\rho$
(d) $\phi_0 = 5$ —Continued						
1.947	0.2923	0.5993	0.1773	0.3736	0.614×10^{-4}	67.4
2.451	.2943	.6372	.1881	.3800	$.107 \times 10^{-3}$	79.2
2.750	.2945	.6538	.1937	.3827	$.142 \times 10^{-3}$	85.9
3.086	.2926	.6662	.2002	.3859	$.181 \times 10^{-3}$	94.1
3.885	.2896	.6888	.2120	.3906	$.278 \times 10^{-3}$	110
6.157	.2782	.7188	.2388	.4001	$.587 \times 10^{-3}$	154
8.698	0.2655	0.7353	0.2628	0.4057	0.947×10^{-3}	203
(e) $\phi_0 = 10$						
0.8900	0.2830	0.4468	0.1077	0.4441	0.698×10^{-5}	34.6
0.7070	.3837	.6912	.1020	.4318	$.529 \times 10^{-4}$	30.2
0.7070	0.4263	0.7820	0.1018	0.4316	0.164×10^{-3}	30.1
(f) $\phi_0 = 20$						
0.8900	0.2512	0.3505	0.08810	0.6968	0.328×10^{-6}	33.8
0.7070	.3208	.5214	.08224	.6665	$.103 \times 10^{-5}$	28.8
0.5616	.3602	.5925	.07845	.6620	$.177 \times 10^{-5}$	25.2
0.4461	0.3984	0.6588	0.07528	0.6556	0.299×10^{-5}	22.5

TABLE 2
PHYSICAL CHARACTERISTICS OF THE COMPOSITE MODEL
FOR $x=\frac{1}{2}$ AND VARIOUS VALUES OF ϕ_0

$Q \times 10^3$	ξ_i	ψ_i	$\mu_e T^{0.25} R / R_\odot$	$\mu_e^2 T^{-0.75} M / M_\odot$	$\mu_e^3 T^{-1.25} L / L_\odot$	$\rho_c / \bar{\rho}$
(a) $\phi_0 = 0$						
1.120.....	0.0616	0.0202	0.7058	1.319	0.807×10^{-3}	104
1.410.....	.0720	.0448	0.8942	1.584	$.216 \times 10^{-2}$	176
1.547.....	.0731	.0511	0.9465	1.637	$.280 \times 10^{-2}$	203
1.776.....	.0744	.0674	1.116	1.834	$.630 \times 10^{-2}$	297
1.947.....	.0738	.0752	1.222	1.938	$.926 \times 10^{-2}$	370
2.451.....	.0704	.0922	1.562	2.260	$.258 \times 10^{-1}$	660
2.750.....	.0682	.1004	1.790	2.459	$.474 \times 10^{-1}$	914
3.086.....	.0631	.1074	2.274	2.909	.121	158×10
3.086.....	.0451	.0898*	7.597	8.509	$.260 \times 10^2$	202×10^2
2.750.....	.0435	.0775*	10.74	12.38	$.146 \times 10^3$	391×10^2
2.451.....	0.0438	0.0702*	14.46	17.09	0.791×10^3	692×10^2
(b) $\phi_0 = 2$						
1.120.....	0.0624	0.0205	0.3998	0.8641	0.229×10^{-4}	107
1.410.....	.0741	.0463	0.4806	1.035	$.693 \times 10^{-4}$	155
1.547.....	.0758	.0532	0.4984	1.037	$.857 \times 10^{-4}$	173
1.776.....	.0782	.0712	0.5640	1.101	$.167 \times 10^{-3}$	235
1.947.....	.0785	.0800	0.5954	1.110	$.219 \times 10^{-3}$	275
2.451.....	.0768	.0996	0.7003	1.179	$.447 \times 10^{-3}$	421
2.750.....	.0745	.1080	0.7699	1.224	$.659 \times 10^{-3}$	539
3.086.....	.0725	.1190	0.8581	1.279	$.102 \times 10^{-2}$	715
3.885.....	0.0678	0.1363	1.055	1.388	0.236×10^{-2}	122×10
(c) $\phi_0 = 3$						
1.120.....	0.0626	0.0206	0.3371	0.9128	0.728×10^{-5}	96.3
1.410.....	.0750	.0468	.3967	0.9793	$.211 \times 10^{-4}$	146
1.547.....	.0768	.0542	.4110	0.9755	$.257 \times 10^{-4}$	163
1.776.....	.0798	.0728	.4555	1.008	$.471 \times 10^{-4}$	215
1.947.....	.0800	.0818	.4806	1.016	$.606 \times 10^{-4}$	251
2.451.....	.0798	.1034	.5437	1.032	$.110 \times 10^{-3}$	358
2.750.....	.0785	.1134	.5843	1.049	$.153 \times 10^{-3}$	437
3.086.....	.0774	.1258	.6308	1.060	$.215 \times 10^{-3}$	543
3.885.....	0.0734	0.1448	0.7382	1.100	0.407×10^{-3}	840
(d) $\phi_0 = 5$						
1.120.....	0.0627	0.0206	0.2781	1.027	0.150×10^{-5}	94.8
1.410.....	.0761	.0478	.3158	1.028	$.400 \times 10^{-5}$	139
1.547.....	.0782	.0554	.3247	1.018	$.479 \times 10^{-5}$	152
1.776.....	.0818	.0748	.3528	1.022	$.812 \times 10^{-5}$	194
1.947.....	.0826	.0846	.3670	1.012	$.101 \times 10^{-4}$	221
2.451.....	.0838	.1086	.4017	0.9902	$.168 \times 10^{-4}$	296
2.750.....	.0842	.1214	.4195	0.9781	$.218 \times 10^{-4}$	342
3.086.....	.0831	.1343	.4460	0.9727	$.288 \times 10^{-4}$	413
3.885.....	0.0816	0.1582	0.4904	0.9529	0.455×10^{-4}	558

TABLE 2—*Continued*

$Q \times 10^3$	ξ_i	ψ_i	$\mu_e T^{0.25} R/R_\odot$	$\mu_e^2 T^{-0.75} M/M_\odot$	$\mu_e^3 T^{-4.25} L/L_\odot$	$\rho_c/\bar{\rho}$
(e) $\phi_0 = 10$						
1.120.....	0.0629	0.0207	0.2196	1.405	0.154×10^{-6}	92.8
1.410.....	.0771	.0486	.2456	1.385	$.392 \times 10^{-6}$	131
1.547.....	.0792	.0563	.2524	1.365	$.466 \times 10^{-6}$	145
1.776.....	.0836	.0768	.2693	1.340	$.757 \times 10^{-6}$	180
1.947.....	.0848	.0872	.2773	1.310	$.905 \times 10^{-6}$	200
2.451.....	.0872	.1135	.2956	1.238	$.142 \times 10^{-5}$	256
2.750.....	.0885	.1280	.3046	1.209	$.178 \times 10^{-5}$	287
3.086.....	.0884	.1431	.3171	1.175	$.221 \times 10^{-5}$	334
3.885.....	.0886	.1710	.3366	1.113	$.323 \times 10^{-5}$	421
6.157.....	.0878	.2244	.3722	1.006	$.600 \times 10^{-5}$	630
8.698.....	0.0868	0.2718	0.4022	0.9289	0.976×10^{-5}	862
(f) $\phi_0 = 20$						
1.120.....	0.0630	0.0209	0.1813	2.202	0.161×10^{-7}	93.5
1.410.....	.0778	.0492	.2014	2.175	$.407 \times 10^{-7}$	130
1.547.....	.0796	.0568	.2067	2.136	$.487 \times 10^{-7}$	143
1.776.....	.0844	.0776	.2188	2.079	$.757 \times 10^{-7}$	174
1.947.....	.0854	.0880	.2256	2.028	$.905 \times 10^{-7}$	195
2.451.....	.0885	.1152	.2382	1.905	$.138 \times 10^{-6}$	245
2.750.....	.0900	.1302	.2444	1.851	$.172 \times 10^{-6}$	272
3.086.....	.0901	.1460	.2530	1.783	$.215 \times 10^{-6}$	313
3.885.....	.0910	.1757	.2658	1.674	$.299 \times 10^{-6}$	386
6.157.....	.0925	.2356	.2847	1.470	$.518 \times 10^{-6}$	545
8.698.....	0.0942	0.2903	0.2971	1.330	0.841×10^{-6}	681

6. *Conclusions.*—An examination of the tables reveals that when $\mu_c/\mu_e = 1$ these composite configurations have radii, masses, and luminosities smaller than those of the sun, while the ratio $\rho_c/\bar{\rho}$ is higher. For $\mu_c/\mu_e = 2$ the radii and masses increase, but the radii in most cases still remain less than that of the sun; the masses and ratio $\rho_c/\bar{\rho}$ are larger than the sun, while the luminosity remains much smaller.

With the data given in Tables 1 and 2 we can derive the course of evolution of a given mass as the degree of degeneracy at the center varies. Thus it is found that $M/M_\odot = 0.1264 \times 10^{-3} \mu_e^{-5} T^{3/4}$ is included among the solutions found with $\phi_0 = 0, 2, 3$, and 5 and for $x = 1$. Interpolating in Table 1, we obtain the results given in Table 4; from this table we can follow the variation in mass and radius for a given central temperature. Table 5 is also derived from Table 3 and shows the variation in temperature and radius for a given mass. Tables 3, 4, 5, 6, 7, and 8 are for $x = 1$, while Tables 9, 10, 11, 12, 13, and 14 show similar results for $x = \frac{1}{2}$.

It will be seen that our values lie far below those given by Gamow and Keller¹³ even if we allow for the difference in κ_0 .

In a later paper we shall combine the results of this and the earlier investigations¹⁴ for a discussion of stellar evolution.

¹³ *Op. cit.*, Fig. 8.

¹⁴ *Ap. J.*, 100, 343, 1944; 102, 216, 1945.

TABLE 3

PHYSICAL CHARACTERISTICS FOR A STAR OF MASS

$$M/M_{\odot} = 0.1264 \times 10^{-5} \mu_e^{-2} T^{3/4} \quad (T \text{ IN DEGREES})$$

$$(T = 2 \times 10^7; x = 1; \mu_e^2 M/M_{\odot} = 0.378)$$

ϕ_0	$\mu_e R/R_{\odot}$	ψ_i	ξ_i	$L\mu_e^3/L_{\odot}$
0.....	0.366			
2.....	.256	0.2567	0.1909	0.507×10^{-3}
3.....	.233	.3994	.2256	$.630 \times 10^{-3}$
5.....	0.185	0.6277	0.2938	0.942×10^{-4}

TABLE 4

MASS RADIUS RELATIONS FOR STARS OF DIFFERENT MASSES
WITH GIVEN CENTRAL TEMPERATURE AND $x=1$

(Derived from Table 3)

T	10^8	5×10^8	10×10^8
$\mu_e^2 M/M_{\odot}$	0.040	0.134	0.225
ϕ_0	$\mu_e R/R_{\odot}$	$\mu_e R/R_{\odot}$	$\mu_e R/R_{\odot}$
0.....	0.775	0.518	0.436
2.....	.541	.362	.304
3.....	.492	.329	.277
5.....	0.392	0.262	0.220

TABLE 5

MASS RADIUS RELATIONS FOR STARS OF DIFFERENT CENTRAL
TEMPERATURES AND GIVEN MASS AND FOR $x=1$

(Derived from Table 3)

T	73.1×10^6	8.55×10^6	3.39×10^6
M	$M_{\odot}$	$\frac{1}{5} M_{\odot}$	$\frac{1}{10} M_{\odot}$
ϕ_0	$\mu_e R/R_{\odot}$	$\mu_e R/R_{\odot}$	$\mu_e R/R_{\odot}$
0.....	0.265	0.453	0.571
2.....	.185	.316	.399
3.....	.168	.288	.362
5.....	0.134	0.229	0.289

TABLE 6

PHYSICAL CHARACTERISTICS FOR A STAR OF MASS

$$M/M_{\odot} = 0.1598 \times 10^{-5} \mu_e^{-2} T^{3/4} \quad (T \text{ IN DEGREES})$$

$$(T = 2 \times 10^7; x = 1; \mu_e^2 M/M_{\odot} = 0.478)$$

ϕ_0	$\mu_e R/R_{\odot}$	ψ_i	ξ_i	$L\mu_e^3/L_{\odot}$
0.....	0.397	0.1172	0.1466	0.246×10^{-2}
2.....	.378	.4362	.2024	$.736 \times 10^{-2}$
3.....	0.432	0.5787	0.1976	0.223×10^{-1}

TABLE 7

MASS RADIUS RELATIONS FOR STARS OF DIFFERENT MASSES
WITH GIVEN CENTRAL TEMPERATURE AND $x=1$

(Derived from Table 6)

T	10^6	5×10^6	10×10^6
$\mu_e^2 M/M_\odot$	0.050	0.169	0.284
ϕ_0	$\mu_e R/R_\odot$	$\mu_e R/R_\odot$	$\mu_e R/R_\odot$
0.....	0.840	0.562	0.472
2.....	.800	.535	.450
3.....	0.913	0.610	0.513

TABLE 8

MASS RADIUS RELATIONS FOR STARS OF DIFFERENT CENTRAL
TEMPERATURES AND GIVEN MASS AND FOR $x=1$

(Derived from Table 6)

T	53.6×10^6	6.26×10^6	2.49×10^6
M	$M_\odot$	$\frac{1}{5} M_\odot$	$\frac{1}{10} M_\odot$
ϕ_0	$\mu_e R/R_\odot$	$\mu_e R/R_\odot$	$\mu_e R/R_\odot$
0.....	0.311	0.531	0.669
2.....	.296	.506	.637
3.....	0.338	0.577	0.727

TABLE 9

PHYSICAL CHARACTERISTICS FOR A STAR OF MASS

$$M/M_\odot = 0.4584 \times 10^{-5} \mu_e^{-2} T^{3/4} \quad (T \text{ IN DEGREES})$$

$$(T = 2 \times 10^7; x = \frac{1}{2}; \mu_e^2 M/M_\odot = 1.371)$$

ϕ_0	$\mu_e R/R_\odot$	ψ_i	ξ_i	$L\mu_e^3/L_\odot$
0.....	0.740	0.0247	0.0635	0.105×10^{-2}
2.....	1.026	.1338	.0685	$.217 \times 10^{-2}$
10.....	0.254	.0588	.0797	$.492 \times 10^{-6}$
20.....	0.293	0.2732	0.0937	0.735×10^{-6}

TABLE 10
MASS RADIUS RELATIONS FOR STARS OF DIFFERENT MASSES
WITH GIVEN CENTRAL TEMPERATURE AND $x=\frac{1}{2}$
(Derived from Table 9)

T	10^6	5×10^6	10×10^6
$\mu_e^2 M/M_\odot$	0.145	0.485	0.815
ϕ_0	$\mu_e R/R_\odot$	$\mu_e R/R_\odot$	$\mu_e R/R_\odot$
0.....	1.565	1.046	0.880
2.....	2.170	1.451	1.220
10.....	0.538	0.360	0.302
20.....	0.620	0.414	0.349

TABLE 11
MASS RADIUS RELATIONS FOR STARS OF DIFFERENT CENTRAL
TEMPERATURES AND GIVEN MASS AND FOR $x' = \frac{1}{2}$
(Derived from Table 9)

T	15.1×10^6	1.55×10^6	0.609×10^6
M	$M_\odot$	$\frac{1}{5} M_\odot$	$\frac{1}{10} M_\odot$
ϕ_0	$\mu_e R/R_\odot$	$\mu_e R/R_\odot$	$\mu_e R/R_\odot$
0.....	0.822	1.402	1.771
2.....	1.140	1.945	2.456
10.....	0.283	0.482	0.609
20.....	0.326	0.556	0.702

TABLE 12
PHYSICAL CHARACTERISTICS FOR A STAR OF MASS
 $M/M_\odot = 0.3320 \times 10^{-5} \mu_e^{-2} T^{3/4}$ (T IN DEGREES)
($T = 2 \times 10^7$; $x = \frac{1}{2}$; $\mu_e^2 M/M_\odot = 0.993$)

ϕ_0	$\mu_e R/R_\odot$	ψ_i	ξ_i	$L\mu_e^3/L_\odot$
2.....	0.456	0.0384	0.0705	0.550×10^{-4}
3.....	.421	.0585	.0777	$.307 \times 10^{-4}$
5.....	.399	.1068	.0837	$.163 \times 10^{-4}$
10.....	0.379	0.2348	0.0876	0.685×10^{-5}

TABLE 13

MASS RADIUS RELATIONS FOR STARS OF DIFFERENT MASSES
WITH GIVEN CENTRAL TEMPERATURE AND $x = \frac{1}{2}$

(Derived from Table 12)

T	10^8	5×10^8	10×10^8
$\mu_e^2 M / M_\odot$	0.105	0.351	0.590
ϕ_0	$\mu_e R / R_\odot$	$\mu_e R / R_\odot$	$\mu_e R / R_\odot$
2.....	0.964	0.644	0.542
3.....	.891	.596	.501
5.....	.844	.564	.474
10.....	0.801	0.536	0.450

TABLE 14

MASS RADIUS RELATIONS FOR STARS OF DIFFERENT CENTRAL
TEMPERATURES AND GIVEN MASS AND FOR $x = \frac{1}{2}$

(Derived from Table 12)

T	20.2×10^8	2.36×10^8	0.937×10^8
M	$M_\odot$	$\frac{1}{6} M_\odot$	$\frac{1}{10} M_\odot$
ϕ_0	$\mu_e R / R_\odot$	$\mu_e R / R_\odot$	$\mu_e R / R_\odot$
2.....	0.455	0.778	0.980
3.....	.420	.719	.906
5.....	.398	.681	.858
10.....	0.378	0.646	0.814

I wish to express my sincere thanks to Dr. S. Chandrasekhar for suggesting this problem and for many helpful discussions.

STELLAR MODELS WITH ISOTHERMAL CORES AND POINT-SOURCE ENVELOPES

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STELLAR MODELS WITH ISOTHERMAL CORES AND POINT-SOURCE ENVELOPES

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ABSTRACT

Stellar models, consisting of isothermal cores and point-source envelopes, are considered, in which the ratio of the mean molecular weight in the core to the mean molecular weight in the envelope varies from 1 to 2. It is found that the upper and lower limit of the mass and radius of the core is a decreasing function of μ_c/μ_e . The evolution of main-sequence stars consequent to the burning of hydrogen in the central regions is also examined.

1. *Introduction.*—The construction of stellar models formed by an isothermal core surrounded by a point-source envelope was discussed by L. R. Henrich and S. Chandrasekhar¹ and by M. Schönberg and S. Chandrasekhar² for two values of the ratio μ_c/μ_e , namely, $\mu_c/\mu_e = 1$ and $\mu_c/\mu_e = 2$. It is now proposed to determine the effect of intermediate values on the fraction of the mass and radius inclosed in the convective core, the total radius and luminosity of the configuration, and the ratio of the central to the mean density.

2. *The equations of the models.*—The equation of state of the isothermal core with negligible radiation pressure is

$$P = K_2 \rho, \quad (1)$$

where

$$K_2 = \frac{k}{\mu H} T. \quad (2)$$

With the foregoing equation of state, the equation of equilibrium,

$$\frac{1}{r^2} \frac{d}{dr} \left(\frac{r^2}{\rho} \frac{dP}{dr} \right) = -4\pi G \rho, \quad (3)$$

can be reduced to the form

$$\frac{1}{\eta^2} \frac{d}{d\eta} \left(\eta^2 \frac{d\phi}{d\eta} \right) = e^{-\phi}, \quad (4)$$

where

$$r = \beta \eta; \quad \rho = \lambda_2 e^{-\phi}; \quad \beta = \left[\frac{K_2}{4\pi G \lambda_2} \right]^{1/2}, \quad (5)$$

and λ_2 is an arbitrary constant.

The mass of the material inclosed up to a radius η is given by

$$M(\eta) = 4\pi\beta^3\lambda_2\eta^2 \frac{d\phi}{d\eta}. \quad (6)$$

The equations of equilibrium for the point-source envelope are used in the standard form given in a recent paper.³

¹ *A. J.*, **94**, 525, 1941.

² *A. J.*, **96**, 161, 1942.

³ Harrison, *A. J.*, **103**, 196, 1946.

3. *The equations of fit.*—Let the interface occur at $\xi = \xi_i$ and $\eta = \eta_i$. Our equations of fit then are

$$-\frac{2}{3}M \frac{\xi^2}{\sigma} \frac{d(\sigma\theta)}{d\xi} = 4\pi\beta^3\lambda_2\eta^2 \frac{d\phi}{d\eta}, \quad (7)$$

$$R\xi = \beta\eta, \quad (8)$$

$$\rho_0\sigma = \lambda_2 e^{-\phi} x, \quad (9)$$

and

$$\frac{k}{\mu H} \rho_0 T_0 \sigma \theta = \frac{k}{\mu H} T \lambda_2 e^{-\phi}, \quad (10)$$

where, in equation (9)

$$x = \frac{\rho_{\text{env}}}{\rho_{\text{core}}} \sim \frac{\mu_{\text{env}}}{\mu_{\text{core}}}. \quad (11)$$

This system of equations can be reduced to one involving only two homology invariant combinations of σ , θ , and their derivatives σ' , θ' , and ξ . If we raise equation (8) to the third power, multiply by equation (9), and divide by equation (7), we are left with

$$-\frac{5}{2} \frac{\sigma^2 \xi}{(\sigma\theta)'} = \frac{\eta e^{-\phi}}{\phi'} x, \quad (12)$$

which we re-write in the form

$$-\frac{5}{2} \frac{\sigma^2 \xi}{(\sigma\theta)'} = u(\xi) = U(\eta) = \frac{\eta e^{-\phi}}{\phi'} x. \quad (13)$$

When we divide equation (7) by equation (10), multiply by equation (9), and divide by equation (8), we have

$$-\frac{2}{5} \frac{(\sigma\theta)'}{\sigma\theta} \xi = \frac{2}{3} \eta \phi' x, \quad (14)$$

which we re-write in the form

$$-\frac{2}{5} \frac{(\sigma\theta)'}{\sigma\theta} \xi = v(\xi) = V(\eta) = \frac{2}{3} \eta \phi' x. \quad (15)$$

4. *The construction of the models.*—Our fitting conditions are those which are usually used, namely, that the mass, radius, and pressure are continuous at the interface; and we have also made allowance for the difference in the mean molecular weights of the core and envelope by making the density discontinuous at the interface (cf. eq. [11]). The physical characteristics of the models are determined by the method used in the previous investigation with the same two sets of integrations of the point-source envelope.⁴

5. *The physical characteristics of the models.*—From equation (15) we readily obtain the formula for the radius R ,

$$R = \tau \frac{\mu_c H}{k} \frac{GM}{T_c}, \quad (16)$$

where

$$\tau = \frac{1}{v_i} \frac{2}{5} \frac{\psi_i \mu_c}{\xi_i \mu_c}. \quad (17)$$

The variation of R will thus be governed only by the factor $\tau\mu_c$ if T_c is assumed to remain constant. This factor is tabulated in Table 1.

⁴ Harrison, *op. cit.*, p. 198.

TABLE 1
PHYSICAL CHARACTERISTICS OF THE COMPOSITE MODEL
FOR VARIOUS VALUES OF x

$Q \times 10^3$	ξ_i	ψ_i	$\tau \mu_c$	$\frac{L_0}{(\tau \mu_c)^{\frac{1}{2}}} \times 10^{-25}$	$\rho_c / \bar{\rho} \times 10^{-2}$
(a) $x = 1$					
1.410....	0.1591	0.1777	0.7743	0.8863	0.805
1.547....	.1612	.1937	0.7802	0.9681	0.977
1.776....	.1660	.2315	0.7857	1.108	1.14
1.947....	.1674	.2491	0.7936	1.209	1.31
2.451....	.1673	.2857	0.8181	1.498	1.84
2.750....	.1656	.3001	0.8381	1.662	5.35
3.086....	.1622	.3144	0.8626	1.837	6.47
3.885....	.1588	.3426	0.9124	2.249	7.81
6.157....	.1445	.3673	1.025	3.362	8.79
8.698....	.1224	.3682	1.199	4.391	27.6
8.698....	.0867	.2716*	1.496	3.931	552
6.157....	.0919	.2341*	1.355	2.924	190 $\times 10$
6.157....	0.1074	0.2720*	1.239	3.058	287 $\times 10^3$
(b) $x = 1/1.2$					
1.547....	0.1214	0.1111	0.8920	0.9053	0.991
1.776....	.1250	.1390	0.9078	1.031	1.31
1.947....	.1264	.1540	0.9193	1.122	1.51
2.451....	.1268	.1851	0.9589	1.384	2.15
2.750....	.1260	.2000	0.9829	1.535	2.60
3.086....	.1237	.2146	1.016	1.693	3.30
3.885....	.1193	.2380	1.079	2.069	5.07
6.157....	.1061	.2685	1.243	3.053	13.2
8.698....	0.0860	0.2702	1.500	3.926	53.4
(c) $x = 1/1.4$					
1.776....	0.1032	0.1023	0.984	0.990	1.66
1.947....	.1040	.1139	1.002	1.075	1.93
2.451....	.1036	.1394	1.056	1.319	2.86
2.750....	.1023	.1515	1.089	1.457	3.56
3.086....	.0999	.1614	1.130	1.606	4.47
3.885....	.0948	.1832	1.216	1.948	7.82
3.885....	.0596	.1251*	1.497	1.756	180 $\times 10$
3.885....	0.0703	0.1401*	1.401	1.815	106 $\times 10^3$
(d) $x = 1/1.6$					
1.776....	0.0896	0.0838	1.036	0.964	2.11
1.947....	.0897	.0932	1.060	1.045	2.53
2.451....	.0883	.1149	1.127	1.277	4.00
2.750....	.0867	.1251	1.169	1.406	5.20
3.086....	.0835	.1350	1.225	1.542	7.25
3.885....	.0769	.1503	1.341	1.855	14.5
3.885....	.0520	.1160*	1.585	1.706	520
3.086....	.0536	.0972*	1.463	1.410	166 $\times 10$
3.086....	0.0624	0.1064*	1.378	1.454	958 $\times 10^2$

*This intersection corresponds to one for which ψ_i has already reached its maximum and thus is not a physically reliable solution.

TABLE 1—*Continued*

$Q \times 10^3$	ξ_i	ψ_i	$\tau\mu_c$	$\frac{L_0}{(\tau\mu_c)^{\frac{1}{2}}} \times 10^{-25}$	$\rho_c/\bar{\rho} \times 10^{-2}$
(e) $x=1/1.8$					
1.947....	0.0795	0.0811	1.106	1.024	3.36
2.451....	.0770	.0998	1.189	1.243	5.82
2.750....	.0738	.1072	1.250	1.360	8.16
3.086....	.0705	.1164	1.313	1.489	12.8
3.086....	.0484	.0924*	1.516	1.386	451
2.750....	.0485	.0813*	1.458	1.260	111×10
2.451....	.0509	.0747*	1.378	1.155	247×10
2.451....	0.0569	0.0793*	1.324	1.178	258×10 ²
(f) $x=1/2$					
1.776....	0.0720	0.0654	1.115	0.930	3.57
1.947....	.0714	.0730	1.149	1.004	4.54
2.451....	.0677	.0893	1.247	1.213	9.01
2.750....	.0638	.0955	1.319	1.324	14.3
3.086....	.0569	.1006	1.429	1.427	36.3
3.086....	.0501	.0940*	1.498	1.394	105
2.750....	.0454	.0790*	1.493	1.244	360
2.451....	0.0454	0.0712*	1.442	1.128	698

For a homologous family of stellar configurations based on Kramers' law of opacity, the luminosity is given by a formula of the form

$$L = \frac{L_0}{Q\mu^{7.5}} \frac{M^{5.5}}{R^{0.5}} \mu_e^{7.5} \times 1.809 \times 10^{-28}, \quad (18)$$

where Q is a parameter defined as in a previous paper⁵ and L_0 can be determined from the particular value of Q which satisfies the equations of fit. Combining equations (17) and (18), we have

$$L = \frac{L_0}{(\tau\mu_c)^{0.5}} \left(\frac{k}{GH} \right)^{0.5} \frac{1}{Q\mu^{7.5}} T_c^{0.5} M^5 \mu_e^{7.5} \times 1.809 \times 10^{-28}. \quad (19)$$

It is clear from equation (19) that, if the central temperature is assumed constant, the variation in the luminosity will be governed by the factor $L_0(\tau\mu_c)^{-0.5}$; this factor is tabulated in Table 1.

The ratio of the mean to the central density is also readily found from equations (5) and (6),

$$\frac{\rho_c}{\bar{\rho}} = \frac{\psi_i \eta_i}{3 \xi_i^3 \phi_i'}. \quad (20)$$

This quantity is also tabulated in Table 1.

6. *Conclusions.*—The curves in Figure 1 illustrate the variation of the radius and luminosity of a star for different values of μ_c/μ_e and constant central temperature for the composite model and the generalized Cowling model.⁶ The upper and lower figures on each curve are ψ_i and ξ_i , respectively. It is clear from an inspection of the curves that

⁵ *Ibid.*, p. 196.

⁶ Harrison, *A p. J.*, 96, 343, 1944.

the upper and lower limit of the mass and radius of the core is a decreasing function of μ_c/μ_e ; the maximum value is reached for $\mu_c/\mu_e = 1$ and the minimum value for $\mu_c/\mu_e = 2$.

By using Figure 1 we can trace the evolution of main-sequence stars. During the relatively early stages of the evolution of a star, as a result of hydrogen combustion in the center, there will be either the formation of an isothermal core or a shrinkage of the convective core. In the latter case the evolution would take place along the curve for the generalized Cowling model (also shown in Fig. 1) in the direction of decreas-

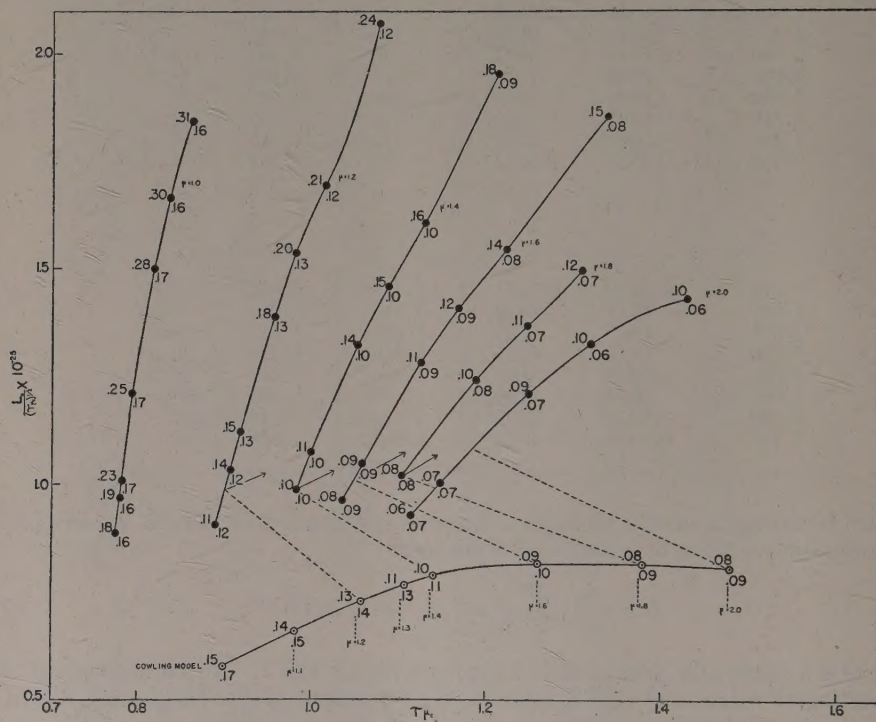


Fig. 1.—The curves illustrate the variation in the luminosity for constant central temperature as a function of the radius for stellar models with isothermal cores and point-source envelopes and the generalized Cowling model. The dotted lines and arrows trace the evolution of main-sequence stars.

ing ψ_i and increasing μ_c/μ_e . The growth of an isothermal core at the center would slow down, and finally stop, convection when the fraction of the stellar mass inside it exceeded a value depending on μ_c/μ_e as shown at the lower end of each curve (●). The transition from one model to another would take place somewhat along the dotted lines as shown in the figure; this transition can take place only at points on the isothermal curves where the mass of the core is equal to the mass of the core of the generalized Cowling model. In the course of the evolution the radius slowly contracts, thus giving rise to an increase in the luminosity which is necessary for the transition. The evolution then continues along a sequence formed by isothermal cores surrounded by point-source envelopes evolving across these curves in the direction of increasing μ_i .

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